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1974 J. Phys. A: Math. Nucl. Gen. 7 1866

(http://iopscience.iop.org/0301-0015/7/15/010)

J. Phys. A: Math., Nucl. Gen., Vol. 7, No. 15, 1974. Printed in Great Britain. © 1974

Einstein's interior field equations in charged elastic media

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Received 5 November 1973, in final form 29 May 1974

Abstract. In a preceding paper, we obtained a solution of the field equations of general relativity for an elastic sphere of constant density. The purpose of the present investigation is to obtain a solution of the charged elastic fluid distribution in general relativity.

1. Introduction

Inside a material medium, the electromagnetic field is represented by the two tensors F_{ij} and M_{ij} which satisfy Maxwell's equations

$$F_{[ij,k]} = 0, \tag{1.1}$$

$$M^{ij}{}_{,j} = J^i, \tag{1.2}$$

J^i being the charged current vector.

The tensors M^{ij} and J^i can be expressed as

$$M^{ij} = \left(\frac{1}{\bar{\mu}} - \bar{\epsilon}\right) F^{[i}_{k}\lambda^{j]}\lambda^k + \frac{1}{\bar{\mu}} F^{ij}, \tag{1.3}$$

and

$$J^i = \sigma F_j{}^i \lambda^j + \rho_e \lambda^i, \tag{1.4}$$

where σ is the conductivity, ρ_e is the space–charge density, λ^i is the four velocity of the medium and $\bar{\epsilon}$ and $\bar{\mu}$ are dielectric constant and magnetic permeability.

The energy–momentum tensor for the electromagnetic field has been assumed to be that of Abraham (1909). Thus

$$E_{ij} = -\tfrac{1}{2}F_{(ir}M^r_{j)} + \tfrac{1}{4}g_{ij}F_{rs}M^{rs} + \tfrac{1}{2}(\bar{\epsilon}\bar{\mu} - 1)F_{ls}\lambda^s\lambda_k(M^k_{(i}\lambda^l\lambda_{j)} + 2M^{kl}\lambda_i\lambda_j). \tag{1.5}$$

We define Einstein's interior field equations for charged bodies by taking a linear superposition of the energy–momentum tensors of the material field and the electromagnetic field, namely

$$R_{ij} - \tfrac{1}{2}Rg_{ij} = -8\pi(T_{ij} + E_{ij}) = -8\pi\tau_{ij}, \tag{1.6}$$

where T_{ij} is the energy–momentum tensor of the elastic fluid distribution given by

$$T_{ij} = -\rho\hat{\lambda}_i\lambda_j + \tfrac{1}{2}C^{kl}_{ij}(\tilde{g}_{kl} - \tilde{g}^0_{kl}), \tag{1.7}$$

where the terms have their usual meaning as in our previous paper (Roy and Singh 1973).

2. The boundary conditions

We now consider the possibility of constructing a model consisting of charged matter inside a three-dimensional region with an electrostatic field generated outside. We assume that in the absence of a charge distribution we get a finite model of neutral matter. In the construction of a finite model of charged matter we have to consider two space–time regions, namely the region occupied by the matter and the region outside it. In a model in which there is an abrupt change from matter to vacuum some discontinuity in the smoothness of g_{ij} will occur. Such a discontinuity is overcome by satisfying certain jump conditions across the surface of separation. For this we assume the continuity of the metric potentials and their first derivatives across the surface of separation and the only discontinuity allowed is in the second normal derivatives of the metric potentials. Further we have to impose the reality conditions (Marder 1964)

$$\tau_4^4 > 0 \qquad \text{and} \qquad \tau_i^i > 0 \tag{2.1}$$

inside the material system. In the case of a charged system exactly the same conditions have to be satisfied by the metric potentials and the total energy–momentum tensor. A linear superposition of the energy momentum tensors of the material field and the electromagnetic field is taken to be the representation of the energy–momentum tensor of the composite field.

3. Solutions of the field equations

We now consider the case of a static, spherically symmetric distribution of charged fluid which forms a linear isotropic medium. The space–time representing the fluid is spherically symmetric. Its line element can therefore be put in the form

$$ds^2 = e^{\alpha}\, dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2) - e^{\beta}\, dt^2, \tag{3.1}$$

where α and β are functions of r only.

We also assume that the electromagnetic field is static and spherically symmetric. The non-vanishing components of F_{ij} are F_{14} and F_{23}. Since there is no radial magnetic field we set F_{23} equal to zero. Similarly, the non-vanishing components of M^{ij} are M_{14}. F_{14} and M_{14} are functions of r alone.

From (1.6), we see that

$$G_{12} = G_{13} = G_{23} = G_{14} = G_{24} = G_{34} = 0$$

leads to

$$\lambda_1 = \lambda_2 = \lambda_3 = 0. \tag{3.2}$$

Also,

$$M_{14} = \bar{\epsilon} F_{14}. \tag{3.3}$$

The field equations (1.6) lead to

$$e^{-\alpha}\left(\frac{\beta'}{r} + \frac{1}{r^2}\right) - \frac{1}{r^2} = 4\pi(\nu + 2\mu)(e^{-\alpha} - 1) + 4\pi\bar{\epsilon} e^{-(\alpha+\beta)} F_{14}^2, \tag{3.4}$$

$$e^{-\alpha}\left(\frac{\beta''}{2} - \frac{\alpha'\beta'}{4} + \frac{\beta'^2}{4} + \frac{\beta' - \alpha'}{2r}\right) = 4\pi\nu(1 - e^{\alpha}) - 4\pi\bar{\epsilon} e^{-(\alpha+\beta)} F_{14}^2, \tag{3.5}$$

$$e^{-\alpha}\left(\frac{\alpha'}{r}-\frac{1}{r^2}+\frac{e^{\alpha}}{r^2}\right) = 8\pi\rho+4\pi\bar{\epsilon}e^{-(\alpha+\beta)}F_{14}^2. \tag{3.6}$$

From (1.2), we find that

$$J^1 = J^2 = J^3 = 0. \tag{3.7}$$

Equation (1.4) then requires that $\sigma = 0$. Hence a static spherically symmetric charged fluid is non-conducting. From equations (1.1), (1.2) and (1.4), we get

$$\frac{d}{dr}(\bar{\epsilon}F_{14}r^2\, e^{-(\alpha+\beta)/2}) = \rho_e r^2\, e^{\alpha/2}. \tag{3.8}$$

We now assume that

$$\rho_e = 3A\, e^{-\alpha/2}, \tag{3.9}$$

where A is a constant.

From equations (3.8) and (3.9), we get

$$\bar{\epsilon}F_{14}\, e^{-(\alpha+\beta)/2} = Ar+\frac{B}{r^2} \tag{3.10}$$

B being a constant. Since we require that F_{14} be regular everywhere we set B equal to zero. Hence

$$\bar{\epsilon}F_{14}\, e^{-(\alpha+\beta)/2} = Ar. \tag{3.11}$$

The left-hand side is the physical component of the tensor M_{ij} and as in classical electromagnetic theory it varies as r.

Substituting the value F_{14} in (3.6) and integrating we get

$$e^{-\alpha} = 1-\frac{8\pi}{r}\int \rho r^2\, dr-\frac{4}{5}\pi\frac{A^2}{\bar{\epsilon}}r^4. \tag{3.12}$$

Taking the density ρ to be constant, we have

$$e^{-\alpha} = 1-8\pi\rho\frac{r^2}{3}-\frac{4}{5}\pi\frac{A^2}{\bar{\epsilon}}r^4. \tag{3.13}$$

From equations (3.4) and (3.13), we have

$$\beta' = \frac{8\pi\rho}{3}\frac{r}{1-\frac{8}{3}\pi\rho r^2-\frac{4}{5}\pi(A^2/\bar{\epsilon})r^4}+\frac{24}{5}\pi\frac{A^2}{\bar{\epsilon}}\frac{r^3}{1-\frac{8}{3}\pi\rho r^2-\frac{4}{5}\pi(A^2/\bar{\epsilon})r^4}+4\pi r(\nu+2\mu)(1-e^{\alpha}),$$

which has a solution of the form

$$e^{\beta} = \left(1-\frac{8\pi}{3}\rho r^2-\frac{4}{5}\pi\frac{A^2}{\bar{\epsilon}}r^4\right)\left(\frac{\frac{2}{5}\pi(A^2/\bar{\epsilon})r^2+\frac{2}{3}\pi\rho+[\frac{4}{9}\pi^2\rho^2+\frac{1}{5}\pi(A^2/\bar{\epsilon})]^{1/2}}{\frac{2}{5}\pi(A^2/\bar{\epsilon})r^2+\frac{2}{3}\pi\rho-[\frac{4}{9}\pi^2\rho^2+\frac{1}{5}\pi(A^2/\bar{\epsilon})]^{1/2}}\right)^2 \times \exp\left(-C+\int 4\pi(\nu+2\mu)r(1-e^{\alpha})\, dr\right) \tag{3.14}$$

where

$$C = +\frac{3}{2}+\frac{2}{3}\frac{\pi\rho}{[\frac{4}{9}\pi^2\rho^2+\frac{1}{5}(A^2/\varepsilon)\pi]^{1/2}}.$$

Hence the most general spherically symmetric line element in charged elastic media is given by

$$\begin{aligned} ds^2 = &\left(1-\frac{8\pi}{3}\rho r^2-\frac{4}{5}\pi\frac{A^2}{\varepsilon}r^4\right)^{-1} dr^2+r^2(d\theta^2+\sin^2\theta\, d\phi^2) \\ &-\left(1-\frac{8\pi}{3}\rho r^2-\frac{4}{5}\pi\frac{A^2}{\varepsilon}r^4\right)\left(\frac{\frac{2}{5}\pi(A^2/\varepsilon)r^2+\frac{2}{3}\pi\rho+[\frac{4}{9}\pi^2\rho^2+\frac{1}{5}\pi(A^2/\varepsilon)]^{1/2}}{\frac{2}{5}\pi(A^2/\varepsilon)r^2+\frac{2}{3}\pi\rho-[\frac{4}{9}\pi^2\rho^2+\frac{1}{5}\pi(A^2/\varepsilon)]^{1/2}}\right)^2 \\ &\times \exp\left(-C+\int 4\pi(\nu+2\mu)r(1-e^{\alpha})\, dr\right) dt^2. \end{aligned} \quad (3.15)$$

Equation of state for elasticity tensor C_{hijk} is given by

$$\frac{\partial}{\partial r}\tau_1^1+\frac{2}{r}(\tau_1^1-\tau_2^2)+\frac{\beta'}{2}(\tau_1^1-\tau_2^2) = 0. \quad (3.16)$$

It is to be noted that the strained sphere is in equilibrium, external body forces are zero and hence the pressure terms are functions of r alone. This is in accordance with the classical elasticity theory in which the stresses on an elastic sphere depend on the radius when it is subjected to spherically symmetric deformation.

Outside the fluid sphere the line element is that due to a spherically symmetric charged particle of total charge $4\pi e$, namely

$$ds^2 = \left(1-\frac{2m}{r}+4\pi\frac{e^2}{r^2}\right)^{-1} dr^2+r^2(d\theta^2+\sin^2\theta\, d\phi^2)-\left(1-\frac{2m}{r}+4\pi\frac{e^2}{r^2}\right) dt^2. \quad (3.17)$$

The non-vanishing component of the electromagnetic tensor F'_{ij} for the metric (3.17) is F'_{14} given by

$$F_{14} = \frac{e}{r^2}. \quad (3.18)$$

The total charge $4\pi e$ within the fluid sphere of radius r_1 is given by

$$4\pi e = \int_0^{r_1}\int_0^{\pi}\int_0^{2\pi} 3Ar^2\sin\theta\, dr\, d\theta\, d\phi, \quad (3.19)$$

so that

$$A = \frac{e}{r_1^3}. \quad (3.20)$$

The internal field (3.15) must fit at the boundary with the external field for which we require that g_{ij} and τ_1^1 be continuous at $r = r_1$.

We thus have the three conditions

$$m = \frac{4}{3}\pi r_1^3\rho+\frac{2\pi}{5}\frac{A^2}{\varepsilon}r_1^5+2\pi\frac{e^2}{r_1}, \quad (3.21)$$

$$\nu(r_1)+2\mu(r_1) = 0 \quad (3.22)$$

and

$$2\ln\left(\frac{\frac{2}{5}\pi(A^2/\bar{\epsilon})r_1^2+\frac{2}{3}\pi\rho+[\frac{4}{9}\pi^2\rho^2+\frac{1}{5}\pi(A^2/\bar{\epsilon})]^{1/2}}{\frac{2}{5}\pi(A^2/\bar{\epsilon})r_1^2+\frac{2}{3}\pi\rho-[\frac{4}{9}\pi^2\rho^2+\frac{1}{5}\pi(A^2/\bar{\epsilon})]^{1/2}}\right) = C-4\pi\int_0^{r_1}(\nu+2\mu)r(1-e^{\alpha})\,dr. \tag{3.23}$$

From (3.21) we determine the value of m when the radius and the density are given. Equations (3.22) and (3.23) are two conditions that the function $\nu+2\mu$ has to satisfy at the boundary. When such a function has been chosen the equation (3.16) then uniquely determines the scalars ν and μ.

Since $e^{-\alpha}$ is to be positive for all values of r we require that

$$1-\frac{8\pi\rho}{3}r^2-\frac{4\pi}{5}\frac{A^2}{\bar{\epsilon}}r^4 > 0. \tag{3.24}$$

Hence

$$r_1^4+\frac{10}{3}\frac{\bar{\epsilon}\rho}{A^2}r_1^2 < \frac{5}{4}\frac{\bar{\epsilon}}{\pi A^2}$$

that is

$$r_1^4-\frac{3}{8\pi\rho}r_1^2+\frac{3e^2}{10\bar{\epsilon}\rho} < 0,$$

and

$$\left(r_1^2-\frac{3}{16\pi\rho}\right)^2 < \frac{9}{256\pi^2\rho^2}-\frac{3e^2}{10\bar{\epsilon}\rho}.$$

This leads to

$$e < \left(\frac{15\bar{\epsilon}}{128\rho}\right)^{1/2}\frac{1}{\pi} \tag{3.25}$$

and

$$\frac{3}{16\pi\rho}-\left(\frac{9}{256\pi^2\rho^2}-\frac{3e^2}{10\rho\bar{\epsilon}}\right)^{1/2} < r_1^2 < \frac{3}{16\pi\rho}+\left(\frac{9}{256\pi^2\rho^2}-\frac{3e^2}{10\rho\bar{\epsilon}}\right)^{1/2}. \tag{3.26}$$

The condition (3.26) tells us that for a given material density and charge the radius of the fluid sphere has both an upper and lower bound. The condition (3.25) states that for a given density the total charge has an upper bound.

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